NATIONAL ACHIEVEMENTS IN CONTROL THEORY (THE AEROSPACE PERSPECTIVE)

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Abstract: It is well known that among the first motivations for modern control theory were dynamic optimization problems in rocket launching and navigation in aerospace. These problems had become especially important in the forties and fifties due to requirement to minimize various costly resources and design parameters, such as flight time, amount (mass) of fuel, weight of the spacecraft, the drag forces and other items. This had to be done under various restrictions on control capacities and other complicating factors , such for example, as incomplete information on the system. In the precious funds of applied mathematical techniques there had long been stored an adequate tool for such problems: it is the Calculus of Variations. Problems in flight dynamics had become the earliest serious technical object for its application. A large number of new basic ideas for adapting Calculus of Variations to modern control problems and synthesizing them into modern control theory were elaborated in the course of investigations in flight dynamics. This presentation traces some seminal investigations, which were crucial for related theoretical developments in former Soviet Union and present Russia and had also influenced related research beyond national borders. Such investigations had good historical precursors in the earlier mathematical works of P.L.Chebyshev, A.M.Lyapunov, A.A.Markov, the works in mechanics by N.E.Zhukovski and S.A.Chaplygin and the activities in dynamic systems theory of the thirties (A.A.Andronov, L.S.Pontryagin et al.).

The present paper is confined only to deterministic problems in *trajectory analysis, control and optimization* within the framework of *mathematical theory of controlled processes.* The national community of researchers involved in these topics was enormous, including those in the Academy of Sciences, the Universities and the numerous institutions and plants supervised by related industrial ministries. While giving tribute to all those involved, this paper does not claim to give a full review of available publications, concentrating on what the authors believe to be *the seminal issues in the field and their role in future directions of research.* This publication will therefore inevitably have a subjective flavour. We sincerely apologize to all those whose contributions may have been missed.

1. CONTROL IN FLIGHT DYNAMICS

As mentioned in the above, the early motivations for modern control theory were problems of dynamic optimization in rocket launching and navigation in aerospace. The first mathematical techniques to be used for these problems were sought for within the Calculus of Variations. Indeed, flight dynamics problems seem to be the earliest serious technical object for its application. And indeed, a large number of new basic ideas for adapting Calculus of Variations to problems in control and synthesizing these within a framework for control theory were elaborated precisely in the course of investigations in flight dynamics. However the new applied problems of optimal control for dynamic processes essentially differed from canonical propositions in Calculus of Variations. The equations of motion for aircraft's mass center in an inertial frame of reference have the form

$$m\dot{\mathbf{V}} = \mathbf{R} + G + P, \quad \dot{\mathbf{r}} = \mathbf{V}, \quad \dot{m} = -\beta_f, \quad (1)$$

where r and $\mathbf{V}$, are, respectively, the radius vector and the velocity vector of the aircraft's mass center; m is the mass; β_f is the mass (propellant) consumption per second; $\mathbf{R}$ is the resultant vector of the aerodynamic forces, $\mathbf{P}$ is the engine thrust vector; $\mathbf{G} = mg$ is the aircraft weight; and g is the gravitational acceleration. As a rule, the motion of aircraft is considered in the wind system of coordinates:

$$\begin{aligned}
&\dot{V} = (1/m)[P\cos(u-\theta)\cos\beta - \\
&\quad - X\cos(-\beta) + Z\sin(\beta) - G\sin\theta], \\
&\dot{\theta} = (1/mV)\{[\sin(\alpha-\theta)\cos\gamma + \\
&\quad + \cos(\alpha-\theta)\sin\beta\sin\gamma] - X\sin\beta\sin\gamma + \\
&\quad + Y\cos\gamma - Z\cos\beta\sin\gamma - G\cos\theta\}, \qquad (2) \\
&\dot{\psi} = (1/mV\cos\theta)\{P[\sin(\alpha-\theta)\sin\gamma - \\
&\quad - \cos(\alpha-\theta)\sin\beta\cos\gamma] + X\sin\beta\cos\gamma + \\
&\quad + Y\sin\gamma + Z\cos\beta\cos\gamma\}, \\
&\dot{h} = V\sin\theta, \ \dot{m} = -\beta_f, \ \dot{x} = V\cos\theta
\end{aligned}$$

where V is the velocity, h is the altitude, x — the range on the Earth's surface; θ — the angle of inclination of the trajectory to the local horizon; α — the angle of attack; ψ — the yaw angle; β — the angle of slide slip; γ is the angle of bank; θ — the angle between the thrust vector and the velocity vector; P — the engine thrust, which is a deterministic function of h, V and β_f, X, Y, Z are the aerodynamic drag, the lift, and the side force respectively:

$$\begin{aligned}
X &= c_x(M,\alpha)(\rho(h)V^2/2)S; \\
Y &= c_y(M,\alpha)(\rho(h)V^2/2)S; \qquad (3) \\
Z &= c_z(M,\alpha)(\rho(h)V^2/2)S,
\end{aligned}$$

where $\rho(h)$ is the atmospheric density; S is the effective wing area; M the Mach number, equal to the ratio of the flight velocity to the velocity of sound $a(h)$ at given altitude; the c_x, c_y, c_z are the aerodynamic coefficients.

Equations (1) and (2) relate two essentially different groups of variables. The variables $\mathbf{r}$, $\mathbf{V}$, m with components h, V, θ, ψ, x, z and m in the wind system of coordinates, enter the equations together with their first derivatives and thus characterize the *state* of the process; the number of these variables is equal to the order of the system. The variables α, β, γ, ϕ and β_f enter the equations without their derivatives and thus are the *controls*. Classification of the variables into phase coordinates and control elements is closely linked with the particular choice of mathematical model for the system being controlled. In some problems the mathematical model (system (1)) does not provide a sufficiently accurate description of the actual behaviour of the aircraft, and it can be improved by supplementing it with an equation of the angular motion of the aircraft about its mass center.

The variables α, β, and γ become the phase coordinates, and the rudder deflection angles assume the role of control elements. On the other hand, in some problems, certain phase coordinates may be upgraded to the status of control elements without detrimental effects; this would involve dropping the corresponding differential equations from the mathematical model. This approach was actually applied by some authors in solving the problems of powered ascent of aircraft when the trajectory inclination angle θ was used as a control element. In addition to differential equations, we have to consider a variety of constraints on the variables, which stem from the particular properties of the system being controlled. The following typical constraints are imposed on aircraft flying in the denser atmospheric layers: the altitude $h \geq 0$, the angle of attack $\alpha_{min} \leq \alpha \leq \alpha_{max}$; the dynamic head $q = 1/2\rho V^2 \leq q_{max}$, the total overload $N = [X_2(h, V, a) + Y_2(h, V, a)]1/2G \leq N_{max}$, and the surface temperature $T_w(h, , V, a) \leq T_{w,max}$. These constraints define conditions for actual (current) time $t < T$. Certain additional conditions are also imposed on the initial and final states of the system. For example, a vehicle may be designed to transport a payload from starting point h_0, x^0 on the ground, where it was at rest ($V_0 = 0$) with starting mass m_0, to a circular orbit at assigned altitude h_1 ($\theta_1 = 0$, $V_1 = V_cir$). The latter conditions define the terminal state set $\mathcal{M}$ in (**??**). Examples of optimality criteria for aircraft are

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that the flight range $\int_{\tau}^{\vartheta} V \cos\theta dt$ should achieve maximum, the flight duration $T = \vartheta - \tau$ or fuel consumption $m(\tau) - m(\vartheta)$ should achieve minimum, the final altitude $h(\vartheta)$ or the final velocity $V(\vartheta)$ should achieve maximum.

The early work in this field was due to D. E. Okhotsimsky, T.M.Eneyev, V.A.Egorov and their colleagues (see Okhotsimsky, 1946; Okhotsimsky and Eneyev, 1957; Egorov, 1958), as well as to I. V. Ostoslavskii and A. A. Lebedev (1946). We shall return to their contributions after a tour to basic theory.

2. THEORETICAL ACHIEVEMENTS. CONTROL THEORY. PONTRYAGIN'S MAXIMUM PRINCIPLE

The process of formalizing and analyzing applied problems of control generated an array of new mathematical ideas. In the nineteen-fifties and sixties this let to the initiation of a new branch of applied mathematics, namely, the "mathematical theory of optimally controlled processes" or, a broader engineering version known simply as "the theory of control".

Loosely speaking, control theory provides us with two basic methods to investigate optimal processes: the Maximum Principle of L.S.Pontryagin — a generalization for nonsmooth functions and constraints of the Euler–Lagrange variation method and the method of Dynamic Programming due to R. Bellman — a generalization of the classical Hamilton–Jacobi method which has been recently propagated to nonsmooth functions as well in the form of generalized "viscosity solutions" and their equivalents. The first method is adequate for the problem of open-loop programmed control while the second method is for the problem of feedback "closed-loop" control synthesis.

We now present a description of Pontryagin's Maximum Principle which was published in a series of pioneering publications (Pontryagin, 1958; Pontryagin *et al.*, 1962), and at the ICM (International Congress of Mathematicians) in 1958 (Edinburgh) and at the First IFAC Congress in Moscow in 1960.

Pontryagin's Maximum Principle is a proposition which gives relations for solving the variational problem of *optimal open-loop control.* In general this is a *nonclassical variational problem* which allows to treat functions and constraints that are beyond those considered in classical theory, but are very natural for applied problems.

The Maximum Principle was formulated in 1956 by L. S. Pontryagin, and further developed by himself and his associates V. G. Boltyanski, R. V. Gamkrelidze, E. F. Mischenko followed by many other investigations. It was motivated by new problems in automation and aerospace engineering, initiating the "mathematical theory of controlled processes". The maximum principle was and is broadly used for solving applied problems of control and other problems of dynamic optimization. It has triggered numerous generalizations and applications. The basic necessary conditions from classical Calculus of Variations follow from the Maximum Principle. In many Western publications the Maximum Principle of Pontryagin is also referred to as the "the Minimum Principle" (By changing signs in some of the upcoming relations the "maximum condition" of the sequel may be rewritten in the form of a minimum condition).

We now proceed with a more detailed formulation.

The Typical Problem of Open-Loop Control. One of the typical problems of optimal open-loop control is as follows. Given are the vector-valued equations of system model

$$dx/dt = f(x, u), \tag{4}$$

where $x \in \mathbb{R}^n$ is the n-dimensional *state* of the system and $u \in \mathbb{R}^m$ — the m-dimensional *control.* Also given are the *starting point* x^0 and the *terminal point* $x^{(1)}$:

$$x(0) = x^0, \quad x(\tau) = x^{(1)}. \tag{5}$$

Relations (**??**) are the *boundary conditions.* The range of the control is the *constraint set* $\mathcal{P}$.

Problem OOLC of Optimal Open-Loop Control is to find such a function $u(t)$ which would steer the system from starting point x^0 to terminal point $x(\tau) = x^{(1)}$ under constraint

$$u(t) \in \mathcal{P}, \tag{6}$$

while minimizing an integral *cost functional*

$$J = J(u(\cdot), x(\cdot)) = \int_{\tau}^{\vartheta} f_0(t, x(t), u(t))\, \mathrm{d}t \tag{7}$$

of (**??**), with $\varphi(x) \equiv 0$, under boundary conditions (5).

Here $x(\cdot)$ stands for the entire function $x(t)$, $t \in [0, \tau]$. The terminal point $x(\tau) = x^{(1)}$ may be substituted by a terminal *target set* $\mathcal{M}$ and the time τ may be either fixed or free. Note that the term "cost functional" is usually applied to problems of minimization while for maximization problems the term "performance index" is more common. The fact that the control $u = u(t)$ is selected among functions of time t indicates that we have the problem of *open-loop control.*

The problem in which $f_0(x, u) \equiv 1$ and the time τ is free brings us to $J = \tau = \min_u$. This is the *time-optimal control* problem where one has to select a control $u(t)$ which steers the system from x^0 to $x^{(1)}$ in minimal time.

We shall say that $u(t)$, $x(t)$, is *a pair* if $u(t)$ is the control which generates the trajectory $x(t)$ of (**??**), with $x(0) = x^0$.

The solution to the Problem of OOLC is the pair $\{u^0(t), x^0(t)\}$, where $u^0(t)$ is the *optimal control* and $x^0(t)$ — the *optimal trajectory.* Clearly, the optimal control must satisfy the constraint (6) and the optimal trajectory must satisfy the boundary condition (5). The pair $\{u^0(t), x^0(t)\}$ must minimize the cost criterion (7). (In this article we always presume that our problem is to *minimize* J. If necessary to *maximize* J, we just have to minimize $-J$).

Loosely speaking, *Pontryagin's Maximum Principle gives the necessary conditions for a control $u^0(t)$ to be the OOLC.*

The Maximum Principle. We will now formulate the local necessary conditions of optimality for the OOLC Problem — Pontryagin's Maximum Principle.

Consider a scalar function (the *Hamiltonian*)

$$\mathcal{H}(\Psi, x, u) = \psi_0 f_0(x, u) + (\psi, f(x, u))$$

of the variables Ψ, x, u, where $\Psi = \{\psi_0, \psi\} \in \mathbb{R}^{(n+1)}$, $\psi \in \mathbb{R}^n$, (p, q) is the scalar product of p, q.

Once $\mathcal{H}(\Psi, x, u)$ is given, it is possible to assign to this function a related system of ODE's (ordinary differential equations)

$$\frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial \psi}, \quad \frac{d\psi}{dt} = -\frac{\partial \mathcal{H}}{\partial x}, \tag{8}$$

or, in more detail,

$$\frac{dx_i}{dt} = \frac{\partial \mathcal{H}}{\partial \psi_i}, \quad \frac{d\psi_i}{dt} = -\frac{\partial \mathcal{H}}{\partial x_i}, \quad (i = 1, \dots, n).$$

Note that the first equation in (2) is (**??**) (!). This is the so-called *canonical Hamiltonian system.*

Here, for a given *pair* $u(t)$, $x(t)$, the second system of (2) should be treated as follows

$$\frac{d\psi}{dt} = -\frac{\partial \mathcal{H}(\Psi, x(t), u(t))}{\partial x}, \tag{9}$$

Main Theorem (The Maximum Principle). Suppose $u^0(t)$ is an open-loop control which steers system (**??**) *from $x^0 = x(0)$ to $x^{(1)} = x(\tau)$, while $x^0(t)$ is the respective trajectory. In order that the pair $u^0(t)$, $x^0(t)$ would optimize the cost functional* (1), *it is necessary that there would exist a constant $\psi_0 \leq 0$ and a solution $\psi(t)$, to the system* (3), *such that the vector-function $\Psi(t) = \{\psi_0, \psi(t)\} \neq 0$ for all $t \in [0, \tau]$ and for all such t the control $u^0(t)$ would satisfy* **the maximum condition**

$$\max\{\mathcal{H}(\Psi(t), x^0(t), u) \mid u \in \mathcal{P}\} = \\ = \mathcal{H}(\Psi(t), x^0(t), u^0(t)) \equiv 0. \tag{10}$$

The time-optimal problem has a simpler formulation.

The Time-Optimal Control Problem. Other Performance Indices. The time-optimal problem of OOLC is to steer the system (4), (6) from $x(0) = x^0$ to $x(\tau) = x^{(1)}$ in minimal time. As indicated in the above, for the time-optimal problem the function $f_0(t, x) \equiv 1$ and the time τ is free. Then there is no need of multiplier ψ_0. The Hamiltonian $\mathcal{H} = H$ now looks like

$$H(\psi, x, u) = (\psi, f(x, u)) = \sum_{i=1}^{n} \psi_i f_i(x, u),$$

and the equation (9) like

$$\frac{d\psi_i}{dt} = -\frac{\partial H(\psi, x(t), u(t))}{\partial x_i}, \quad i = \{1, \dots, n\} \tag{11}$$

Remark 3.1 The maximum principle is far more simpler for linear systems where under additional conditions of *controllability* type it is also a sufficient condition of optimality. A complete theory of linear controlled systems was developed by N.N.Krasovski (1968) using the techniques of functional analysis.

Interpretations and generalizations of the Maximum Principle. The maximum principle is closely related to *classical calculus of variations*, but is a step forward in the direction of treating nonsmooth functions and nonclassical constraints which are very common in engineering and other types of applied problems. The connections with classical calculus and the differences that arise are discussed in publications given in the bibliography.

Among the generalizations of the Maximum Principle are those directed on more complicated nonsmooth constraints — state constraints, mixed constraints on the controls and trajectories, functional and nonclassical integral constraints, etc. The treatment of these problems is usually based on the techniques of nonlinear analysis and "nondifferentiable" dynamic optimization. Here the functions f, f_0, ϕ and the other functions involved in the formulation of the problem are allowed to be nondifferentiable. The adjoint equations may have multivalued right-hand sides, turning into *differential inclusions* and the numerical solution schemes are more complicated than in the standard case. Grasping the techniques of these generalizations requires special knowledge which may lie beyond the scope of traditional engineering mathematics.

A special topic are the conditions when the Maximum Principle is *sufficient* for optimality in the general case. This question is closely related to the connections between the Maximum principle and Dynamic Programming . Other special classes of OOLC problems are those when the solution leads to *singular controls*. Such controls appear when the Maximum Principle degenerates ($H(x, u, \psi) \equiv 0$) and they have to be found through additional procedures involving *higher order necessary conditions*. (The topics of the present paragraph lines are discussed below in Section 4).

The Maximum Principle was also propagated to *discrete systems*, *systems with distributed parameters*, systems with *after-effects including time-delays*, and other types of infinite-dimensional systems (J-L. Lions in France, A. G. Butkovski, Yu. V. Egorov, A. I. Egorov et al. in USSR). It was also modified for *stochastic systems*. (One should note however, that for stochastic systems a realistic and practically applicable setting is the one of *feedback control* or *Closed-Loop Optimal Control*, effectively treated within the techniques of *Dynamic Programming*.)

Rather general classes of variational problems with non-classical constraints (including non-strict inequalities) or with nonsmooth cost functionals are used to be called *problems of Pontryagin type*. The demand for solving such problems stimulated new research in differential equations and differential inclusions, nonlinear analysis and extremal problems, numerical methods and other related domains.

Among the significant achievements in developing modern-type variational methods for nonsmooth problems were those suggested by A. A. Dubovitski and A. A. Milyutin (1965), as well as by V. F. Demianov (1972) and B. N. Pschenichniy (1971) in Kiev, and their associates. A propagation of classical calculus of variations to non-classical problems was developed by V. A. Troitskii (1971). Active research on optimal control was carried out in Minsk by R. F. Gabasov and F. M. Kirillova.

On numerical methods A crucial element for effectively applying the Maximum Principle as well as other techniques are numerical methods and reliable software. Among the first numerical procedures for the Maximum Principle were the algorithms developed by I. A. Krylov and F. L. Chernousko (1972). Very significant contributions in numerical techniques and algorithmic developments belong to R. P. Fedorenko (1964) of the Institute of Applied Mathematics and Yu.G.Evtushenko of the Computing Center and their colleagues at the Russian Academy of Sciences. A considerable number of applied algorithms was developed at institutions and plants within the system of related industrial ministries. At the same time original solutions and numerical methods were developed by A. E. Bryson, N. J. Kelley, G. Leitmann, E. Polak and their colleagues in USA.

3. THEORETICAL ACHIEVEMENTS. DEVELOPMENTS IN DYNAMIC PROGRAMMING

The method of Dynamic Programming (DP) is attributed to Professor R. Bellman of Rand Corporation and University of South California at Los Angeles (Bellman, 1957). Its continuous-time version is a very broad generalization of classical Hamilton–Jacobi techniques to variational problems of control. The method is connected with embedding the construction of the optimal process into a family of identical problems with arbitrary initial conditions. This requires that the *control would depend both on time and the state space variable*, being presented in a feedback ("closed-loop") form. The first indications on engineering solutions to specific problem of control synthesis were given by Flugge–Lotz in Germany, D. W. Bushaw in USA and A. A. Feldbaum (1955) in USSR. In fact, the work of Feldbaum on feedback control for automation served as an applied motivation for the development of the Maximum Principle, as indicated in (Pontryagin, 1958).

In fact, a considerable amount of research was fulfilled by research groups at the Institute of Control Problems in Moscow, particularly, under the leadership of B.N.Petrov who also initiated research on the theory of invariance — the conditions of independence of the system outputs from the inputs (Petrov, 1960). Applied problems of flight control were investigated by A. M. Letov (1969), the first President of IFAC. New results in system identification and adaptive control were introduced by B. T. Polyak and Ya. Z. Tsypkin (1980).

The Value Function. A typical DP problem of control synthesis in continuous time is as follows. Given are system (4), (6) and state constraint $\mathcal{Y}(t)$ (a set-valued function with compact values, continuous in t). One is to find a *value function*

$$V(\tau, x) = \min_u \left\{ \int_\tau^{\vartheta \wedge \sigma} f_0(t, x, u)dt + \right.$$
$$\left. + \varphi_0(\vartheta, x(\vartheta)) \wedge \varphi_1(\sigma, x(\sigma)) \right\}$$

along the trajectories of system (4), (6). Here φ_0, φ_1 are the terminal functions. The trajectory $x(t, \tau, x)$ starts at time τ from point $x \in \operatorname{int} \mathcal{Y}(t)$ and σ is the first instant of time when it reaches the boundary $\partial\mathcal{Y}(\sigma)$ of set $\mathcal{Y}(\sigma)$. Then, if $\sigma < \vartheta$, the process stops and the integration ends at $t = \sigma$ with terminal function $\varphi_1(\sigma, x(\sigma))$. Otherwise the integration ends at fixed time ϑ with terminal function $\varphi_0(\vartheta, x(\vartheta))$.

A sufficient condition for $V(\tau, x)$ to be the value function is to satisfy the next HJB (Hamilton–Jacobi–Bellman) equation

$$V_t + \min\{(V_x, f(t,x,u)) + f_0(t,x,u)\} = 0$$

under boundary conditions

$$V(\vartheta, x) = \varphi_0(\vartheta, x), \text{ if } x(t) \in int\mathcal{Y}(t),\ t \in [\tau, \vartheta),$$

or $V(\sigma, x) = \varphi_1(\sigma, x)$, if $\sigma < \vartheta$, $x(t) \in \operatorname{int}\mathcal{Y}(t)$, $t \in [\tau, \sigma)$, $x(\sigma) \in \partial\mathcal{Y}(\sigma)$.

The solution to the control problem in this setting is given by a function $u^0 = u^0(t,x)$ of *both time and state* which yields a *synthesized strategy* defined for any feasible "position" $\{t, x\}$. In order that the problem would be solved it is sufficient, beside solving the HJB equation, to ensure that system $\dot{x} = f(t, x, u^0(t,x))$ does have a solution in some appropriate sense.

The first general solution to a control synthesis problem was the the one of "linear-quadratic" control — on minimizing a quadratic integral cost functional for linear systems. It was broadly known as the "R. Kalman — A. M. Letov" solution. Various versions of control problems with quadratic cost were due to A. I. Lur'ye, V. A. Yakubovich et al. in Leningrad (now St.Petersburg) and to N. N. Krasovski et al. in Sverdlovsk (now Yekatherinburg).

However, at the early stages of development, this essential approach was mostly not feasible for continuous-time systems. This is due to the fact that for most applied problems with constraints given by inequalities, the value function $V(t,x)$ turned out to be nondifferentiable. The difficulty was surpassed by applying new results in nonlinear analysis which led to he introduction of generalized, so-called "viscosity" solutions. The initiating ideas for these (due to professors O. A. Oleinik (1957) and S. N. Kruzhkov (1970) of Moscow State University) were developed in final form by professors P-L. Lions (France), M. G. Crandall and L. C. Evans (USA) and their colleagues M. Bardi (Italy) and H. Ishii (Japan). An independent equivalent "minmax" form of solution was introduced by by A. I. Subbotin (Russia) (1995). The generalized "nonsmooth" theory of Dynamic Programming is also among the basic tools for developing the theory of game-theoretic problems of dynamics which allows to treat problems of guidance and trajectory tracking under disturbances of various kind.

The achievements in developing continuous-time Dynamic Programming allowed to formalize an array of problems on control synthesis under various types of state constraints and obstacle problems (see papers by A. B. Kurzhanski and Yu. S. Osipov (1968; 1969)). These results are relevant for example, for automatic guidance of unmanned aircraft in mountainous terrain, for calculating safety zones in motion planning and related problems (Gusev and Kurzhanski, 1971; Kurzhanski, 2004).

Once the value function is known, one may construct a *backward reachability set* $W(t) = \{x : V(t,x) \leq \mu$ for a given μ which ensures that $W(t)$ is nonempty.

The Problem of Control Synthesis. Given set W find control strategy $u(t,x)$ $(U(t,x))$ that steers system (4) from any position $\{\tau, x\}$, $x \in W(t)$ ensuring $V(t,x) \leq \mu$.

Then each of the strategies $u(t,x)$ $(U(t,x))$ may be sought for directly, through a unified scheme. Namely, considering function $\mathcal{V} = d^2(x, W(t))$, introduce either the single-valued strategies

$$u^0(t,x) \in \mathcal{U}(t,x) = \arg\min\{ \\ \exp(-2\lambda t)(\mathcal{V}_x(t,x), f(t,x,u)) | u \in \mathcal{P}(t)\},$$

$\lambda > 0$, or the set-valued strategies $U^0(t,x) = \mathcal{U}(t,x)$, depending on the type of system and the definition of solutions used. (Here λ is the Lipschitz constant in $\{t,x\}$ for function f).

The problem here is that the proposed strategy $u^0(t,x)$ or $U^0(t,x)$ must satisfy in some appropriate sense the equation

$$\dot{x} = f(t, x, u^0(t,x)), \tag{12}$$

or the differential inclusion

$$\dot{x} \in f(t, x, U^0(t,x)). \tag{13}$$

For a general nonlinear system of type (12) the solution may be defined as a "constructive motion" introduced in (Krasovski, 1970), (Krasovski and Subbotin, 1998), while in case of linear systems ($f(t,x,u) = A(t)x + B(t)u$) with convex compact constraints on the controls the solutions $U^0(t,x)$ may be taken in the class of upper semicontinuous set-valued strategies with synthesized system (13) treated as a differential inclusion (Krasovski, 1964). The last move may lead to sliding regimes and chattering control. Solutions involving chattering controls were discussed in detail in (Zelikin and Borisov, 1994).

Guaranteed state estimation. The generalization of DP also allowed to develop a coherent nonstochastic theory of *guaranteed identification and state estimation* introduced simultaneously in USA (Witsenhausen in 1968, F.Schweppe in 1968 and 72) and USSR (N. N. Krasovski (1968), A. B. Kurzhanski (1970; 1977)). In applied form similar methods had been actively developed and used in guidance of spacecraft vehicles (M. L. Lidov (1971; 1984), I. A. Boguslavski (1970), P. E. Elyasberg et al. (1980)). Further developments were due to F. L. Chernousko (1994), V. M. Kuntsevich (1992), and B. T. Polyak (1980), with numerical techniques emphasized in publications (Kurzhanski and Vályi, 1997; Kostousova, 2001).

Remark The seminal papers of A. N. Kolmogorov (1941) and N. Wiener (1949) on extrapolation and interpolation of stationary time series led to the introduction of the widely known "Wiener filter". A further step was made by R. Kalman (Kalman, 1960) who had introduced a separate equation for the measurement device and produced a system of differential equations for the (now celebrated) "Kalman filter" which in closed form gave a recurrent description of the linear-quadratic estimate for systems corrupted by Gaussian noise. His related work on controllability and observability was also announced at the first IFAC Congress in Moscow in 1960 (Kalman, 1960). However a considerable number of problems in control, navigation and related areas are such that the disturbances and uncertainties in the model and system inputs do not allow any statistical description, being unknown but bounded with given bounds.This created a demand for an estimation theory under information assumptions other than in the Wiener–Kalman filters. This precisely was *the theory of guaranteed or set-membership state estimation.*

The theory of guaranteed state estimation was further developed for many classes of applied problems (Milanese, 1995). It is a cornerstone for treating control problems under *incomplete measurement information.*

The theory of Dynamic Programming is also among the basic tools for developing the theory of game-theoretic problems of dynamics which allows to treat problems of guidance and trajectory tracking under disturbances of various kind. These will be discussed below in Section 6. But before passing to the latter we shall first discuss some topics closely related to the Dynamic Programming approach.

4. OTHER NEW IDEAS. SUFFICIENCY CONDITIONS FOR OPTIMALITY. SOLVING DEGENERATE PROBLEMS

Throughout the advent of the Maximum Principle and the Dynamic Programming there appeared some other new ideas based on sufficient conditions of global optimality for control processes and related mathematical techniques. The main instrument of this approach is the so-called bounding function $\mathcal{V}(t,x)$. Its assignment yields the following constructions:

$$\begin{aligned}
&R(t,x,u) = \partial\mathcal{V}/\partial t+\\
&\quad + (\partial\mathcal{V}/\partial x, f(t,x,u)) - f^0(t,x,u);\\
&G(x) = F(x) + \mathcal{V}(\vartheta, x);\\
&\zeta^0(\mathcal{V}) = (u^*0(t), x^*(t)) =\\
&\quad = \arg\max\{R(t,x,u)|(u,x)\in D(t)\},\ t\in A;\\
&l(\mathcal{V}) = \min_x G(x) - \mathcal{V}(\tau, x_\tau)-\\
&\quad - \int_\tau^\vartheta \max\{R(t,x,u)|(u,x)\in D(t)\}dt;\\
&u^*(t,x) = \arg\max_u R(t,x,u); P(t,x) =\\
&\quad \max_u R(t,x,u);\\
&\Delta(\mathcal{V}) = \int_\tau^\vartheta [\max_x P(t,x) - \min_x P(t,x)]dt+\\
&\quad + \max_x G(x) - \min_x G(x),
\end{aligned}$$

where $f(t,x,u)$ is the right-hand side of equation (4); $f_0(t,x,u)$, $\varphi(\vartheta,x)$ are the integrand and terminal function in the optimality criterion (5); (f,v) stands for the inner Euclidean product and $\sigma = \vartheta$.

Sufficient Conditions of Optimality. The process $\zeta^*(\mathcal{V})$ is an optimal process,$\zeta^*(\mathcal{V}) = (u^0(t), x^0(t)) = \zeta^*$, when it is an admissible process: $\zeta^*(\cdot) = (u^*(\cdot), x^*(\cdot)) \in \mathcal{D}$ and *Proposition A* is true:

$$\begin{aligned}
\zeta^0(u^0(t), x^0(t)) = \zeta^*(\mathcal{V}) = (u^*(\cdot), x^*(\cdot)) =\\
= \max\{R(t,x,u)|(u,x)\in D(t)\}.
\end{aligned}$$

A bounding function $\mathcal{V}$ is further named to be *the solving function.* Note that if a controlled process $\zeta(t) = \{u(t), x(t)\} \in D(t)$ is *admissible* means it must satisfy the constraints on controls u and states x, whether preassigned or given on-line.

In particular, the necessary conditions for optimality — the equality $u = u^*(\mathcal{V})$, are as follows:

$$R_x(t, x^0(t), u^0(t)) = 0,$$

$$R(t, x^0(t), u^0(t)) = \max\{R(t, x^0(t), u)|u \in \mathcal{D}(x)\}$$

are the equations of Pontryagin's maximum principle. Pontryagin's adjoint vector-function ψ coincides with the gradient of function $\mathcal{V}$ along the trajectory $x^0(t)$, namely, (see notations of Section 1) $\partial\mathcal{V}(t, x^0(t))/\partial x = \psi(t)$.

For any bounding function $\mathcal{V}(t,x)$ the functional $l(\mathcal{V})$ is a lower bound of the functional J on $\mathcal{D}$ and $J(\zeta^0) = \min_\zeta J(\zeta) = \max_\mathcal{V} l(\mathcal{V})$.

A closed-loop control $u^*(t,x)$ is approximately optimal with an estimate

$$J(u^*(t,x), x^*(\cdot)) - \min_u J(u(t,x), x(\cdot)) \le \Delta(\mathcal{V}),$$

$\forall\tau$, x_τ, for any function $\mathcal{V}$.

The closed-loop control $u(t,x) = u^*(t,x)$ is optimal and the bounding function $\mathcal{V}$ is the Bellman "value function" $V(t,x)$, when $P(t,x) = 0$, $G(x) = 0$. We then come to the Hamilton–Jacobi–Bellman (HJB) equation. The key idea of this approach is a total decomposition of the controlled optimization problem with respect to time. This functional problem is reduced to a parametric family of elementary problems on maximizing the function of state and control $R(t, x, u)$ with t being a parameter. The optimal trajectory $x^0(t)$ and the control programm $u^0(t)$ are defined by the equalities of Proposition A.

Within this maximization the control and state (u, x) are free of the equations of system dynamics. A process $\zeta^*(\varphi)$ is defined with the accuracy of function $\mathcal{V}(t, x)$, which is selected such that the result of $(u^*(t), x^*(t))$ of this maximization satisfies the dynamic equations (4). The methods of such selection are the main issue in the given approach.

The solutions through the method of variations which yield conditions for only a local minimum, are actually somewhat incomplete as compared with the present approach. The logical scheme of the present method also does not require the existence of a desired optimal process in some a priori fixed class of functions unlike the method of variations. The global method under discussion made possible to find and provide necessary mathematical techniques for new classes of solutions to variational problems with rather exotic properties which however are typical for many applied issues. The Sufficient Condition of Optimality keeps the function $\mathcal{V}$ undefined. Its additional definition through various methods is precisely the essential part of the solutions and presents a broad field for new inventions and using new methods. In particular, it produces effective techniques for solving the so-called degenerate problems where neither the method of variations nor the Hamilton–Jacobi method may be sufficiently effective. Considerations of simplicity and clarity play an important role in formulating the basics of the approach. However these may lead to a loss of their generality. At the same time a fundamental question does arise which is whether a generalization of these sufficiency conditions would allow to make them both necessary and sufficient. At this moment there are some positive answers to this question.

The techniques and approaches of this section are due to V. F. Krotov (1962) (see Krotov, 1996; Krotov and Gurman, 1973). An interested reader may find the discussion of related results in indicated references.

The earlier ideas already include elements of the presented approach. Thus the method of Lagrange's multipliers as applied to the equations of the controlled process may be interpreted as the first method of decomposition. However, this method is applicable to the problem of absolute minimum only when the lower bound $l(\mathcal{V})$ is defined for $\mathcal{V} = \psi(t)x$. The Hamilton–Jacobi method may also can be considered as an application of special "solving functions". It is a clear-cut method for assigning function $\mathcal{V}$, though it may not cover all possible applications of this theory. The nearest to this theory are the ideas of C. Caratheodory. But methods of introducing solving functions which lie beyond the Hamilton–Jacobi techniques were not under consideration.

Degenerate Problems and Singular Solutions. Degenerate problems are those barely solvable by regular methods of control theory. There may be two main causes for that: (i) an optimal solution may be absent in the "ordinary" class of admissible processes, while the existence of the latter is necessary for applying regular methods and (ii) the equations of the maximum principle may happen to be degenerate and therefore do not produce enough information for obtaining the solution.

In case (i) the optimal solution may be found as a minimizing sequence. We consider here two types of such sequences: those with infinite accumulation of control switchings (the so-called sliding regimes), and those with discontinuous trajectories, including the case of infinite accumulation of discontinuity points. There are three ways to complete the optimization model over the sliding regimes. The first is to introduce a generalized process of the "flow" type (L.C.Young,J.L.Rubio). The second is a reduction of the variational problem (in the form of minimizing terminal cost):

$$\varphi(x(\vartheta)) = \min;\ \dot{x} = f(t, x, u);\ u \in U,$$

to a "relaxed" problem:

$$\varphi(x(\vartheta)) = \min;\ \dot{x} = \mathcal{F}(t, x, u);\ u \in U,$$

where $\mathcal{F}(t, x)$ is the closed convex hull of set $F(t, x) = f[t, x, U]$ (Filippov, 1959; Gamkrelidze, 1969; Ioffe and Tikhomirov, 1974). The third approach is based on the technique of the Sufficient Conditions for Optimality. As opposed to the other two, this approach may be applied not only to sliding regimes but also to discontinuous solutions, as well as to more complicated minimizing sequences. The related theory of discontinuous or degenerate solutions and sliding regimes was elaborated by (Krotov, 1996; Krotov and Gurman, 1973; Dykhta, 1979).

5. VARIATIONAL PROBLEMS IN FLIGHT DYNAMICS

The first of such problems was stated by G.Hammel in 1927: it was to find the control program for rocket engine thrust during a vertical ascent in the atmosphere to achieve a maximum final altitude. A complete solution of this problem was obtained by D. E. Okhotsimskii (1946). Here the controlled process was the motion of the rocket within given time interval $[\tau, \vartheta]$ with the state space variables h, V, m and control variable β_f. This result was significant for mathematical theory of control as well as for an understanding of the specifics of optimal trajectories of booster-rockets. The problem has some features which are not typical at all for Calculus of Variations, but are very typical for modern problems with constraints on the controlling functions. The author here proposed a new method of "direct investigation of variation" for such problems. The optimal trajectory, is synthesized from the following regime sequences: a motion with maximal admissible value of engine thrust; an ascent, realizing an optimal function $V(h)$, and a free ascent with $P = 0$.

In paper of D.E.Okhotsimskii and T.M.Eneyev (1957) the problem of optimal launch of the first sputnik satellite and optimal selection of the stages of the booster-rocket was considered. These papers, together with the investigation of V. A. Egorov (1958) on optimal rocket trajectories, formed the primary amount of knowledge for understanding the synthesis of trajectories and the necessary parameters of the booster-rocket for the first sputnik satellite at the modelling level.

Many interesting papers, devoted to problems in flight dynamics, have been published in journals and other sources that are not readily accessible.

We further consider a degenerate problem of the optimal powered ascent of an aircraft which played an important role in the theory of optimal flight dynamics. The process being controlled here is the motion of an aircraft with constant mass throughout time interval $[\tau, \vartheta]$ with constant mass, state space variables h, V and control variable θ from an initial state to a dynamic state. I. V. Ostoslavski and A. A. Lebedev (1946) obtained an optimal function $V(h)$ (subsonic), realized for an aircraft ascent. Independently, A. Miele later found the second, supersonic function $V(h)$ and proved that the optimal trajectory is synthesized with either minimal or maximal admissible values of the angle θ.

Sufficient optimality conditions introduced, by V. F. Krotov and M. M. Khrustalev specify this synthesis uniquely, having developed the results under following presumptions: the aircraft mass is variable, available are two control variables — the angle θ and the thrust parameter β_f. There is a possibility of considering the problem within a fixed range and with arbitrary $g(h)$.Therewith, a principal construction of the optimal trajectory is maintained and is completed with the equality: $\beta_f = max$. Such an expansion allows us to construct analytically the optimal trajectories for the wide range of the rockets and the aircraft. It is interest to compare this solution with a result D.E.Okhotsimski: the mass consumption β_f must be variable to realize the optimal function $V(h)$. But this fact does not coincide with the solution described here: $\beta_f = \max = constant$, but the optimal function $V(h)$ is maintained with the second control parameter — the angle θ.

The basic technique for optimal synthesis of aircraft trajectories in setting (1), (2) is Pontryagin's Maximum Principle described below. However, the hidden presence of degenerate problems of the above requires additional procedures, since the optimal solutions may not be unique; the adjoint variable ψ of the maximum principle may be insufficient for finding the solution; the absence of robustness may require techniques of regularizing respective numerical schemes. All these elements,if coped with, will finally yield a qualitative structure of the optimal trajectory and additional means for numerical solution.

A large number of investigations on trajectories for entering the atmosphere of the Earth and other planets within the setting of (2), (3), as well as within reasonably simplified settings, allowed to develop an understanding of how to synthesize algorithms for the descent of spacecraft, (see Okhotsimskii and Golubev, 1975; Yaroshevski, 1988; Okhotsimsky, 1964). The scope of this presentation does not allow us to discuss the last problem in worthy detail as well as the those of optimal control of interorbital transitions and other space maneuvres (Gurman, 1966).

At this point we would like to mention the exceptional role of the Keldysh Institute of Applied Mathematics in developing control algorithms for space research. A review of these achievements is given in reference (Popov and Akim, 2003).

6. CONTROL SYNTHESIS UNDER UNCERTAINTY

A considerable amount of work was done by the control community in coping with uncertainty and incomplete information in the field of control.

The adopted scheme is based on constructing superpositions of value functions for open-loop control problems. In the limit these relations reflect *the Principle of Optimality* under set-membership uncertainty. This principle then allows one to describe the closed loop reach set as a level set for the solution to the *forward HJBI (Hamilton–Jacobi–Bellman–Isaacs) equation*. The final results are then presented either in terms of value functions for this equation or in terms of set-valued relations.

Schemes of such type have been used in synthesizing solution strategies for guaranteed control, dynamic games and related problems of feedback control under realistic data, including those of control *under incomplete information* and *measurement feedback control*. The control schemes were constructed in backward time in more or less equivalent forms of solvability sets, stable bridges of N. N. Krasovski (1968; 1998), the alternated integrals of L. S. Pontryagin (1980), the scheme of B. N. Pschenichniy in Kiev (Pschenichniy and Ostapenko, 1992). Effective and original dynamic programming-type constructions were developed in USA by R. Isaacs, P. Varaiya, G.Leitmann, R. J. Elliot and N. J. Kalton, T.Basar and others, as well as in France, by A. Blaquiere and later, through the notion of H^∞-control by P.Bernhard and the idea of capture basins by J. P. Aubin and his associates P.Saint-Pierre, M.Quincampoix and others.

Uncertain dynamics. The Standard Model. This is given by differential equation

$$\dot{x} = f(t, x, u, v), \tag{14}$$

with properties of continuous function f defined as in control theory, with inputs representing controls u to be specified and unknown disturbances (v). Here $x \in \mathbb{R}^n$ as always is the *state* and $u \in \mathbb{R}^p$ is the *control* that may be selected either as an *open loop control* — OLC — a measurable function of time t, restricted by the inclusion

$$u(t) \in \mathcal{P}(t), \quad a.e.,$$

or as a *closed-loop control* — CLC — a *feedback strategy* which is either sought for either as a multi valued map

$$u = \mathcal{U}(t, x) \subseteq \mathcal{P}(t).$$

or as a single-valued function $u(t, x) \in U_c$ which ensure existence in some appropriate sense of solutions to differential inclusion

$$\dot{x} \in f(t, x, U(t, x), v), \tag{15}$$

or to differential equation

$$\dot{x} = f(t, x, u(t, x), v) \tag{16}$$

respectively.

Here $v \in \mathbb{R}^q$ is the unknown *input disturbance* with values

$$v(t) \in \mathcal{Q}(t), \quad a.e. \tag{17}$$

$\mathcal{P}(t)$, $\mathcal{Q}(t)$ are set-valued continuous functions with compact values, ($\mathcal{P} \in \operatorname{comp} \mathbb{R}^p$, $\mathcal{Q} \in \operatorname{comp} \mathbb{R}^q$). Given also is a closed target set $\mathcal{M}$.

The Problem of Control Synthesis under Uncertainty is to find an admissible feedback control strategy $\mathcal{U} = u(t, x)$ or $\mathcal{U} = U(t, x)$ which steers system (14) to reach the target set $\mathcal{M}$ despite the unknown disturbances v. Such problems are usually treated within the notions of game-type dynamics introduced by R. Isaacs.

Typical admissible classes of feedback controls and trajectory solutions involved for the given problem are due to N. N. Krasovski in the single-valued case (Krasovski, 1968; Krasovski and Subbotin, 1998) and A. F. Filippov in the multivalued case (Filippov, 1959). It is N. N. Krasovski who introduced the most developed formalized and integrated solution theory for problems in "game-type" controlled dynamics, which was developed further by him and his associates for a broad class of control problems under uncertainty or conflict.

Continuing with the last problem, we are to find the value function

$$V(t, x) = \min_{\mathcal{U}} \max_{x(\cdot)} \{ d^2(x[t_1], \mathcal{M}) \mid \mathcal{U} \in U_c, \ x(\cdot) \in \mathcal{X}_{\mathcal{U}} \}.$$

Here $\mathcal{X}_{\mathcal{U}}$ is the variety of all trajectories of equation (15) or (16).

The formal solution equation for the problem is the Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation

$$V_t + \min_u \max_v (V_x, f(t, x, u, v)) = 0, \tag{18}$$

with minmax often assumed interchangeable and with control $u(t, x)$ to be found from the solution to the minmax problem in (18). However in reality this is just a symbolic relation as the function $V(t, x)$ in general turns out to be nondifferentiable.

The solutions to the control synthesis problem under uncertainty are then found, for example, through procedures of constructing solvability tubes in the form of stable bridges

$$W[t] = \{x : V(t, x) \leq 0\},$$

as introduced by N. N. Krasovski with control strategy further found from conditions of "extremal aiming":

$$u(t, x) = \arg \min_u \{ \max_v d(x, W[t]) \},$$

and trajectories interpreted as "constructive motions" of (Krasovski and Subbotin, 1998). Here $V(t, x)$ is a generalized solution to equation (18).

Along with the theory, the numerical methods of constructing solutions were developed. Several global successive approximation numerical schemes were proposed as well schemes to approximate the deterministic control problem or game by a stochastic discrete-time process.

The game-theoretic approaches in conjunction with set-valued techniques and new results in nonlinear analysis allowed to formalize basic problems of control under measurement feedback and unknown but bounded disturbances.

Various formalizations and applications of the theory of control under incomplete information may be found in books and papers by N. N. and A. N. Krasovski (1994), A. B. Kurzhanski (1977), Yu. S. Osipov and A. V. Kryazhimski (1995), F. L. Chernousko and A. A. Melikyan (1978), V. M. and A. V. Kuntsevich (2002), B. N. Pschenichniy and V. V. Ostapenko (1992).

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